

Aharonov-Bohm Type Forces Between Magnetic Fluxons

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Abstract

Forces related to A-B phases between fluxons with $\Phi = \alpha\Phi_0$ $\alpha \neq integer$ are discussed. We find a $\alpha^2 \ln(r)$ type interaction screened on a scale λ_s . The forces exist only when the fluxons are actually immersed in the region with non vanishing charge density and are periodic in α . We briefly comment on the problem of observing such forces.

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There is no magnetic field outside an ideal infinitely long and thin solenoid and there is no electromagnetic force per unit length between such parallel solenoids or fluxons. In the following we note that the presence of charged particles between and around the fluxons induces a new type of force between them which is of some theoretical interest. We also briefly comment on the prospects of detecting such interactions.

Let us assume that n_F fluxons $\Phi_1, \Phi_2, \dots, \Phi_{n_F}$ have been introduced at locations $\vec{R}_1, \vec{R}_2, \dots, \vec{R}_{n_F}$ where the wave function of a system of N charges $\Psi^{(0)}(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N)$ is non vanishing. The modification of the Schrödinger equation via $\vec{\partial}_i \rightarrow \vec{\partial}_i + \frac{e}{c}\vec{A}_i$ will then change the wave function

$$\Psi^{(0)}(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) \rightarrow \Psi^{(0)}(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N; \vec{R}_1, \Phi_1, \vec{R}_2, \Phi_2, \dots, \vec{R}_{n_F}, \Phi_{n_F}) \quad (1)$$

and shift the initial ground state energy $E^{(0)}$ to

$$E^{(0)} \rightarrow E^{(0)} + \delta E^{(0)}(\vec{R}_1, \Phi_1, \vec{R}_2, \Phi_2, \dots, \vec{R}_{n_F}, \Phi_{n_F}). \quad (2)$$

This induced energy shift can be viewed as an interaction energy between the fluxons:

$$\delta E^{(0)}(\vec{R}_j, \Phi_j) = W(\vec{R}_j, \Phi_j) \quad (3)$$

The gradients $\nabla_{\vec{R}_i} W$ will then yield forces \vec{F}_i acting on the fluxons Φ_i . To simplify the following we assume that the ground state wave function factorizes:

$$\Psi^{(0)}(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = \psi_{\gamma_1}(\vec{r}_1)\psi_{\gamma_2}(\vec{r}_2)\dots\psi_{\gamma_N}(\vec{r}_N) \quad (4)$$

The ground state energy shift induced by introducing the fluxons is then a sum:

$$\delta E^{(0)}(\vec{R}_j, \Phi_j) = \sum_{i=1}^N \delta E_{\gamma_i}(\vec{R}_j, \Phi_j) \quad (5)$$

over the shifts of the individual N energies E_{γ_i} of the N charges. The latter will now be assumed to be fermions say electrons, and suppose, for simplicity, that the electrons do not interact with each other.

Let us first consider the simplest case of one fluxon $\Phi = \alpha\Phi_0$ introduced in the center of a cylindrical region of radius R . In this geometry the free wave functions will be eigenfunctions of the (\hat{z} component of) angular momentum denoted by l and which normally takes on positive and negative integer values $l = 0, \pm 1, \pm 2, \pm 3, \dots$. If the radial degrees of freedom were frozen then we would have simply $E_{l,r}^{(0)} = \frac{\hbar^2 |l|^2}{2mr^2}$. The introduction of the fluxon effectively shifts up by α all the l values

$$|l| \rightarrow |l| + \alpha \text{ for } l \geq 0, \quad |l| \rightarrow |l| - \alpha \text{ for } l < 0 \quad (6)$$

The *sum* of the energy shifts of the *pair* of levels $l = \pm |l|$ is then

$$\delta E_{|l|,r}^{(0)} = \frac{\hbar^2 \alpha^2}{2mr^2} \quad (7)$$

The total energy shift is found by summing $\delta E_{|l|,r}^{(0)}$ over l and r values. Bohr-Sommerfeld quantization suggests discrete $r = r_n$ orbits which along with the discrete $|l|$ in each annular region yield one state per unit area a_0^2 with a_0 the typical distance between the charged particles so as to fill up the Fermi "circle". This parameter is related to the two dimensional ("one layer") density by $n_2 \sim a_0^{-2}$. Thus[1]

$$W_\alpha^{tot} = \sum_{|l|,n} \frac{\hbar^2 \alpha^2}{2mr_n^2} \quad (8)$$

$$\rightarrow \frac{\alpha^2 n_2 \hbar^2}{2} \int_0^R \frac{2\pi r dr}{r^2} = \frac{\pi \alpha^2 n_2 \hbar^2}{2} \ln(R/a_0) \quad (9)$$

is the total energy required in order to insert one fluxon at the center of a cylinder of radius R . An extra 1/2 factor is due to the fact that there are only half as many ($l = \pm |l|$) pairs. The logarithmic dependence of $W_\alpha^{tot}(R)$ on R is expected on general grounds. Assume that the fluxon is inserted at the center of a cylindrical hole of radius R_{in} , inside a concentric annulus of external radius R_{out} . The minimal substitution in the regular, symmetric, gauge $\vec{A} = \alpha \frac{\phi_0}{r} \hat{e}_\theta$, $\vec{\partial} \rightarrow \vec{\partial} + \frac{e}{c} \vec{A}$, is such that the total energy

$$\sum_\gamma \int |(\vec{\partial} + \frac{e}{c} \vec{A})\psi_\gamma|^2 dx dy \quad (10)$$

remains invariant if we scale $x \rightarrow \lambda x$ and $y \rightarrow \lambda y$, provided we have a homogeneous uniform (two dimensional) density

$$n_2(x, y) = \sum_{\gamma} |\psi_{\gamma}(x, y)|^2 \simeq \text{const.} \quad (11)$$

This implies that $W \simeq \log(R_{out}/R_{in})$. The argument holds also for general domains of overall size R and any shape: only the coefficient in $W(R) \simeq c \ln(R)$ would depend on the dimensionless ratio characterizing the shape.

Equation (8) has a quadratic dependence ($\sim \alpha^2$) on the energy on the flux. Due to the invariance of all topological effects under $\Phi \rightarrow \Phi + n\Phi_0$, the energy is periodic under $\alpha \rightarrow \alpha \pm n$. Also the energetics should be invariant under time reversal which flips magnetic fluxons: $\Phi_i \rightarrow -\Phi_i$. This together with periodicity implies that for $1/2 < \alpha < 1$ the coefficient should become $\sim (1 - \alpha)^2$ (instead of $\sim \alpha^2$ in the $0 < \alpha < 1/2$ interval).

For the special case of semi-fluxons ($\alpha = 1/2$) the Schrödinger equation is invariant under this time reversal operation [2] since the flip ($\Phi_j \rightarrow -\Phi_j$) of a semi fluxon amounts to trivial shifts by $\pm\Phi_0$. The wave function $\Psi^{(0)}(\vec{r}_1, \dots, \vec{r}_N; \vec{R}_j, \Phi_j)$ or the individual wave functions $\psi_{\gamma_i}^{(0)}(\vec{r}_i; \vec{R}_j, \Phi_j)$ can be made real by choosing the singular gauge $A_{\theta} = \delta(y)$, $x > 0$ for a fluxon at the origin. In this case real wave function $\psi(x, y)$ simply jumps along the $x > 0$ ray, $\psi(x, y = +\epsilon) = -\psi(x, y = -\epsilon)$; $x > 0$, consistent with the requisite A-B phase of π picked upon circulating the fluxon. Thus real wave function flips sign an odd number of times along any closed path which encloses only Φ_1 at \vec{R}_1 . Therefore an odd number of continuous strings of zeros emanate from \vec{R}_1 , and from all other fluxons at \vec{R}_j . These "null lines" [3] [2] terminate on the boundary of the system (beyond which $\psi_{\alpha_i}(x, y)$ vanish anyway) or, at the location of another fluxon.

In the case of cylindrical symmetry the initial angular real wave functions ($\sin(|l|\theta)$, $\cos(|l|\theta)$) have each $2|l|$ nodal lines at the origin. The shifts $|l| \rightarrow |l| \pm \frac{1}{2}$, induced by introducing a semi-fluxon at the origin, are equivalent to adding/subtracting

one null line - yielding in both cases an odd - $(2l \pm 1)$, number of null lines as required by the general considerations.

Returning to the main issue let us address next the mutual interaction energies of two semi-fluxons at a distance $|\vec{R}_1 - \vec{R}_2| = a$ inside a uniform medium of charged particles, with constant two dimensional density n_2 give by equation (11), introduced near the origin at the center of a large domain. At a distance r from the origin the vector potentials are: $\vec{A}_i = \frac{1}{2}\Phi_0 \frac{\hat{e}_{\theta_i}}{|\vec{r} - \vec{R}_i|}$ $i = 1, 2$ where \hat{e}_{θ_i} refer to the tangential unit vector with respect to \vec{R}_1, \vec{R}_2 as origins. For $r > a$ $\vec{A}_1 + \vec{A}_2$ add, up to small corrections, to the vector potential of a single quantized fluxon at the origin. Insofar as the topological effect of interest are concerned the latter is just a gauge artifact. Hence we expect that only a region of size of order a around each fluxon and between the fluxons will be affected and consequently that mutual interaction energy behave like $W^{tot}(r)$ of eq. (8) with $r \sim a$:

$$W_{(\alpha_1=1/2, \alpha_2=1/2)}(a) = \xi \frac{\pi}{16} \frac{n_2 \hbar^2}{m} \ln(a/a_0) \quad (12)$$

The numerical factor of order one ξ represents the effects of having a two center system.

This interaction leads then to an attractive force between the two semi-fluxons

$$F_{(1/2, 1/2)}(a) \simeq \frac{\xi \pi}{16} \frac{n_2 \hbar^2}{m} \frac{1}{a} \quad (13)$$

Recalling that W and F represent the effect of "one layer" and n_2 is the two dimensional number density in this layer, we can rewrite the last equation in a more usfull form as

$$\frac{F_{(1/2, 1/2)}(a)}{\text{unit fluxon length}} \simeq \frac{\xi \pi}{16} \frac{n \hbar^2}{m} \frac{1}{a} \quad (14)$$

with n the true three dimensional density of the charges. The coefficient $\xi(\alpha_1, \alpha_2)$ should be periodic in both α_1 and α_2 . For positive α_i the force will be attractive if $\alpha_1 + \alpha_2 > 1/2$. However for $\alpha_1 + \alpha_2 < 1/2$ we expect repulsion: the energy of the joint system with Φ_1, Φ_2 overlapping ($\sim (\alpha_1 + \alpha_2)^2$) exceeds the sum of energies of two separete fluxons: $\sim \alpha_1^2 + \alpha_2^2$.

The topological origin of the forces is clearly manifest by the fact that such forces act only on fluxons which are actually immersed in the charged particle background but is absent for fluxons which are outside this region. Thus if in the example of the concentric cylindrical geometry discussed above, we move the fluxon inside a hole, the energy of the system is unchanged and no force is expected. Since there are no charged particles in the hole we can continue using the same gauge potential $\vec{A}(r) = \frac{\alpha\Phi_0}{r}\hat{e}_\theta$ even when the fluxon is not in the center. The key point is for every path enclosing the fluxon (or fluxons - if there are several fluxons inside the hole) that a charge particle confined to the annular domain can *actually* perform - the AB phase will be the same.

The interaction (12) is quasi - confining ($W(r) \rightarrow \infty$ with $r \rightarrow \infty$) just like the two dimensional coulomb interaction. It is well known that for such cases the system may find it energetically favorable once $R \geq \lambda_s = (\text{screening length})$, to screen the charges (or fluxons in the present case). Indeed such a screening is generated by the circulation of all the charged particles of charge e (for fluxon of $\Phi = \Phi_0/2$ say). The corresponding current density at a distance r is

$$\vec{J}(r) = \frac{\hbar\alpha n}{mr}\hat{e}_\theta \quad (15)$$

The screening of the B field is found from Maxwell's equation:

$$\frac{dB_z^{induced}}{dr} = \frac{1}{c}J_\theta = \frac{\alpha e\hbar n}{mcr} \quad (16)$$

The α in eq. (16) depends on r due to the partial screening of the initial fluxon $\alpha = \alpha(r=0)$ by currents circulating between the origin and r :

$$\alpha(r) = (\alpha\Phi_0 - \int_0^r 2\pi r B_z^{induced}(r) dr) / \Phi_0 \quad (17)$$

The coupled equations (16) (17) yield r profiles for $\alpha(r)$ and $B_z^{induced}(r)$ which are exponentially falling off like $\exp(-r/\lambda_s)$, thus defining λ_s . Approximating $\alpha(r) = \alpha\theta(\lambda_s - r)$

we readily find λ_s from

$$2\pi \int_0^{\lambda_s+\epsilon} r B_z^{induced}(r) = \pi \int_0^{\lambda_s+\epsilon} r^2 \frac{dB_z}{dr} = \quad (18)$$

$$= \frac{\pi}{mc} \frac{e\alpha\hbar\lambda_s^2}{2} = \alpha\Phi_0 = \frac{2\pi\alpha\hbar c}{e} \quad (19)$$

where we used integration by parts, eq. (16), and demanded that the net induced flux exactly cancel $\alpha\Phi_0$. Recalling that $\alpha_{em} = \frac{e^2}{\hbar c} = \frac{\lambda_{comp}}{a_{Bohr}}$ we can write λ_s as:

$$\lambda_s = \frac{2}{\alpha_{em}} a_{Bohr} \left(\frac{a_{Bohr}^{-3}}{n} \right)^{1/2} = 150 \left(\frac{10^{25}}{n} \right)^{1/2} A^0 \quad (20)$$

We considered so far a system of free charged fermions. The induced interactions are actually the same for charged bosons. In this case we need not fill up a Fermi sphere of levels and at $T = 0$ all bosons would be in the same ground state $\psi_0(x, y)$. However from the above derivation each occupied level γ_i contributes equally and the same $W(R)$ emerges when all bosons are in the ground state. This is true also for a superconductor in the Landau-Ginzburg effective charged Higgs model description. We then find an extra energy $\simeq \int d^2r |(\vec{\partial} - \frac{e}{c}\vec{A})\phi(r)|^2 \simeq \alpha^2 \phi_0^2 \ln(R/R_0)$ where ϕ_0 is the order parameter due to an improperly quantized fluxon leading again to a $1/r$ force.

In passing we note that QCD is a nice example of confinement-screening interplay. The spectra of heavy quark-antiquark, $\bar{Q}Q$, system suggests a confining linear $\bar{Q}Q$ potential $V = \sigma R$ at "large" distances and the same is expected to QQ baryons in $SU(2)_c$. Creation of $\bar{q}q$ pairs tends to screen the confining potential - and only exponentially falling Yukawa like potentials exist between physical, color neutral, hadrons.

The QCD quarks with non zero triality (screening the confining interaction between $\bar{Q}Q$) play the role of the electrons which transform non trivially under the " Z_2 " of the fluxon in our example, and generate currents screening the interaction between the semifluxons. The mechanisms for screening and confinement tend to be mutually exclusive: both in QCD

and in our example the screening of charges reduce the long range forces and resulting putative pairing of $Q\bar{Q}$, (two semi-fluxons here). Also $\bar{q}q$'s which are already paired by confinement to triality (and color) singlets will not screen the $Q\bar{Q}$ force. The introduction of the two semi-fluxons will not induce here large scale pairing of the electrons. Yet the mechanism of semi-fluxons confinement may quench the screening currents. This could be the case for the Bose-Einstein condensate example. If the distance between the two fluxons a is smaller then the size of the system R , a null line will form between them[4]. This in turn impedes the circulation of screening currents around Φ_1 or Φ_2 separately.

For conducting mesoscopic rings of sizes of order of microns, voltages and persistent currents related to $W(r)$ and J above have already been observed [5]. Can one detect also the force between fluxons? For this the coherence length for electrons and the screening length λ_s of eq. (20) should both exceed the fluxons separation a . For typical metals $n \simeq 3 \cdot 10^{22}$ and $\lambda_s \simeq 0.2\mu$. Together with the intrinsic requirement following from the general topological argument above that the fluxons be immersed inside the metal this appears to make the measurement of the force of eq. (14) in metals virtually impossible.

Type *II* superconductors with semi-fluxons (integer fluxons in the Copper pair $2e$ charge unit), generated by super currents on a penetration length of scale $\lambda_p \simeq 0.2\mu$, appear more promising. By varying the temperature in the interval $T_c > T > 0$ the density of normal, unpaired, electrons can be controlled. The pattern of normal electron circulation would, on its own, be energetically disfavored because it would generate "forbidden" B fields inside the superconductor for $\lambda_p < r < \lambda_s$ and compensating super-currents should locally cancel it. If this does not completely "freeze the system" the introduction of the semi-fluxons will raise its energy by amounts of order $\frac{W(r)}{\text{unit length}} \simeq n_e \frac{\hbar^2}{2m_e} \ln(r/r_0)$. For $n_e = 10^{19}$, a $1/r$ force of order $10^{-5} - 10^{-4}$ dyne/cm would then operate between fluxons which are $1 - 10\mu$ appart. Such forces may tend to pair semi-fluxons or bend them. It is not inconcievable

that such effects may be detected.

Finally it is amusing to compare the topological force (14) with the Casimir force [6] between two parallel conducting wires $\sim \frac{\hbar c}{a^3}$. The ratio is: $\rho \equiv F_{top}/F_{cas} \sim n\hbar a^2/mc^2 \sim n\lambda_{com}a^2$. For electron systems $n \sim a_0^{-3}$ with a_0 of the order of the Bohr radius. Using $\lambda_{com}/a_0 \sim \alpha_{em}$ we have then $\rho \sim \alpha_{em}a^2/a_0^2$. Since generally $a \gg a_0$ this ratio is very large. The origin of this large ratio is easy to assess. Only vacuum fluctuations (photons) on scales $\lambda \sim a$ contribute to the Casimir force whereas *all* electron modes down to wavelength $\lambda \sim a_0$ contribute equally to the interaction energy and force proposed here.

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